Int. J. Heat Mass Transfer. Vol. 36, No. 5, pp. 1373–1387, 1993
Printed in Great Britain

0017-9310/93 \$6.00+0.00

On the linear stability of mixed and free convection between inclined parallel plates with fixed heat flux boundary conditions

A. S. LAVINE

Mechanical, Aerospace and Nuclear Engineering Department, University of California, Los Angeles, CA 90024–1597, U.S.A.

(*Received* 26 *September* 1990 *and in final form* 19 *May* 1992)

Abstract—This paper concerns the linear stability of mixed or natural convection between infinite parallel plates inclined arbitrarily with respect to gravity. In previous analyses, it has commonly been assumed that the temperature at the boundaries is fixed. In the present work, the heat flux at the boundaries is fixed. For this boundary condition, it is shown that if the temperature gradient parallel to the plates is adverse (temperature increasing downward), the flow is unstable for *any* value of the Rayleigh number. That is, the critical Rayleigh number is zero. This result holds independent of the base velocity and temperature distributions and inclination.

INTRODUCTION

THIS PAPER concerns the linear stability of mixed or natural convection between infinite parallel plates inclined arbitrarily with respect to gravity (see Fig. 1). An attempt has been made to allow the class of flows under consideration to be as general as possible, although there are certain restrictions (as noted in the Problem Statement section). The study is fundamental, and in considering the applicability of the results to real systems it must be recognized that there are idealizations present, such as the infinite extent of the domain. Nonetheless, it is hoped that the conclusions of this study will not only be of fundamental interest, but will also advance our qualitative understanding of low Rayleigh number flows in large aspect ratio ducts, such as might occur in solar collectors or chemical vapor deposition systems, for examples.

In previous stability analyses of natural and mixed convection flows, it has commonly been assumed that the temperature at the boundaries is fixed, that is, the temperature perturbation must be zero at the boundary. The present work concerns the case in which the heat flux at the boundaries is fixed, that is, the gradient of the temperature perturbations (in the direction normal to the plates) is zero at the boundary. For this boundary condition, it is shown that if the temperature gradient parallel to the plates is adverse (i.e. temperature increasing downward), then the flow is unstable (to perturbations of zero wavenumber in the directions parallel to the plates), for *any* value of the Rayleigh number. That is, the critical Rayleigh number is zero. This result holds independent of the base velocity and temperature distributions and inclination. This is in contrast to the case of a fixed temperature boundary condition, for which the critical Rayleigh number is typically nonzero, and is dependent on the details of the base flow and inclination.

There has of course been a great deal of research concerning natural and mixed convection between parallel plates. Catton [1] and Ostrach [2, 3], for example, give extensive reviews of natural convection in enclosures, including between parallel plates. The book by Gershuni and Zhukhovitskii [4] gives extensive coverage of the stability theory for vertical, horizontal, and tilted cases, again mostly for natural convection.

The papers reviewed here will be restricted to those which are particularly relevant to the present work for one of two reasons. The first is that they consider a temperature gradient parallel to the plates. (The temperature gradient parallel to the plates will be called the longitudinal temperature gradient.) The second reason is that they consider the effect of the type of boundary condition for the temperature perturbation.

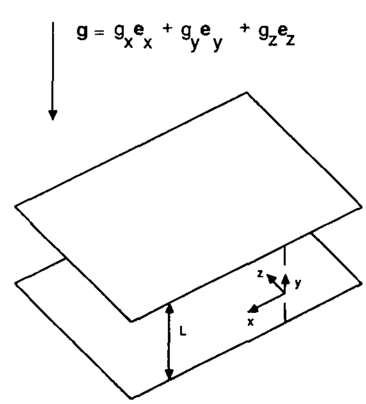

FIG. 1. Parallel plates inclined arbitrarily with respect to gravity.

NOMENCLATURE

A	temperature gradient in x-direction
B	temperature gradient in z-direction
Bi	Biot number, hL_w/k_w
C	amplitude of temperature fluctuations
c	specific heat
$\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$	unit vectors in the coordinate directions
Fo	Fourier number, $\alpha_w/\omega L_w^2$
g	magnitude of gravity vector
$\mathbf{g}$	gravity vector
g_x, g_y, g_z	components of gravity
Gr	Grashof number, $g\beta\gamma L^4/\nu^2$
h	heat transfer coefficient
i	$\sqrt{-1}$
k	wavenumber, $k_x^2+k_z^2$
k_w	wall thermal conductivity
k_x, k_z	wavenumbers
L	channel spacing
L_w	wall thickness
p	pressure
Pr	Prandtl number, ν/α
Ra	Rayleigh number, $Gr\,Pr$
$Ra_{\Delta T}$	Rayleigh number based on temperature difference between plates
Re	Reynolds number, $U_{ch}L/\nu$
T	temperature
t	time
U_{ch}	characteristic velocity
$\mathbf{u}$	velocity vector
u, v, w	velocity components.

Greek symbols

α	thermal diffusivity
β	thermal expansion coefficient
γ	longitudinal temperature gradient, $(g_xA+g_zB)/g$
θ	$\hat{T}/Pr$
λ	see equation (27)
ν	viscosity
ρ	density
$\hat{\sigma}$	growth rate
σ	$\hat{\sigma}\,Re\,Pr$
σ_{max}	maximum value of σ
ψ	$Re(\hat{u}A+\hat{w}B)/\gamma$, or $\psi^2 = Re^2(\hat{u}^2+\hat{w}^2)g^2/(g_x^2+g_z^2)$
ω	frequency of temperature fluctuations.

Subscript

w	wall.

Other symbols

$\bar{}$	base quantity
*	dimensional quantity
$\tilde{}$	perturbation quantity
$\hat{}$	y-variation of perturbation quantity (except in $\hat{\sigma}$).

Longitudinal temperature gradient

Gershuni and Zhukhovitskii [4] reviewed the cases of vertical and inclined plates with a constant vertical temperature gradient. In this situation, mechanical equilibrium exists, that is, the base flow is zero. In the case of a fixed temperature boundary condition, the flow becomes unstable at a finite, positive value of the Rayleigh number (Rayleigh number defined positive for an adverse temperature gradient). Gershuni and Zhukhovitskii also reviewed the case of vertical plates with a vertical temperature gradient *and* a temperature difference between the two plates. In this case, the base flow is nonzero. When there is only a lateral temperature difference, the velocity is cubic, and instability sets in at a certain Grashof number based on the lateral temperature difference. Relative to this case, an adverse temperature gradient is destabilizing, as expected. The mechanism of instability can be either hydrodynamic (due to the increased base velocity caused by the longitudinal temperature gradient) or convective (due directly to the destabilizing effect of the adverse temperature gradient), depending on the values of the governing parameters.

Several authors have considered vertical and inclined plates with a favorable temperature gradient (temperature increasing upward), particularly in the context of natural convection in a slot of finite length. Vest and Arpaci [5] studied the vertical case both analytically and experimentally. In their analysis, however, they neglected the effect of the longitudinal temperature gradient on the base flow, and omitted it from the disturbance energy equation. Hart [6] and Bergholz [7] considered the inclined case, and both discussed the mechanisms of instability, based on the energy integrals.

Nakayama *et al.* [8] considered the case of Poiseuille flow between horizontal plates with a longitudinal temperature gradient, with the bottom plate hotter than the top. (Note that this does *not* meet the definition of an *adverse* longitudinal temperature gradient to be given later in this paper, because the temperature gradient in the plane of the plates is perpendicular to the gravity vector.) It was seen that the longitudinal temperature gradient has the effect of making the flow less stable to longitudinal rolls. Balakrishnan [9] corrected Nakayama's work to include the effect of the longitudinal temperature gradient on the base flow, but found that this does not have a significant effect on the stability results, at least for the range of parameters considered.

Boundary conditions for temperature perturbations

First, let us begin with a discussion of the terminology 'fixed flux' or 'fixed temperature' boundary condition, and the physical realization of these idealized conditions. These terms have been used by other authors, such as Sparrow *et al.* [10]. However, some authors use terms such as 'perfectly insulating boundary' for the fixed flux case, and 'perfectly conducting boundary' for the fixed temperature case (for instance, Gershuni and Zhukhovitskii [4]). The present author prefers the terms 'fixed flux' and 'fixed temperature' because they directly translate into the mathematical boundary conditions for the perturbations. Also, the phrases 'perfectly insulating boundary' and 'perfectly conducting boundary' do not precisely describe the conditions necessary to ensure fixed flux and fixed temperature boundary conditions, respectively. This will now be discussed. (The reader is also referred to Knowles and Gebhart [11], for further discussion concerning the effect of thermal boundary conditions on the stability of external natural convection on a vertical flat plate.)

In any real system, conduction within the walls that bound the fluid affects the boundary condition at the solid/fluid interface. Therefore, let us consider the wall which bounds the flow domain. On the outside of this wall is some mechanism for controlling the wall heat flux or temperature. For instance, there could be a heater outside of the wall, or another flowing fluid. The issue at hand is, if the fluid inside the wall is undergoing temperature perturbations, what is the boundary condition for the temperature perturbations at the wall/fluid interface? In particular, under what conditions can it be approximated by either a fixed flux or a fixed temperature boundary condition?

To answer these questions, let us consider the model problem illustrated in Fig. 2. The wall has thickness L_w and thermal conductivity and diffusivity k_w and α_w. The inside of the wall is exposed to the flow, which can be characterized with a heat transfer coefficient h, and an oscillating fluid temperature, $C \cos \omega t$. The outer wall boundary condition can be specified heat flux or temperature. It is an elementary (although tedious) task to solve for the periodically varying wall temperature distribution, which depends on the Biot number, $Bi = hL_w/k_w$, and the Fourier number based on the period of oscillations, $Fo = \alpha_w/\omega L_w^2$. In each

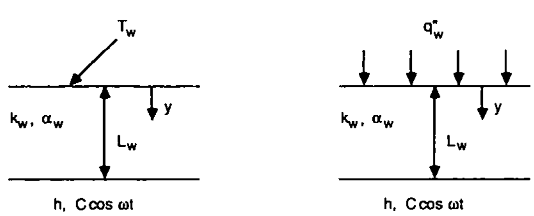

FIG. 2. Model problems for investigating boundary conditions for temperature perturbations.

case then, both the temperature and the heat flux at the wall/fluid interface can be determined. It is seen that in both cases, the temperature at the wall/fluid interface will be nearly constant for small $Bi^2 Fo$, whereas the heat flux at the wall/fluid interface will be nearly constant for large $Bi^2 Fo$. Thus the appropriate boundary condition for the temperature perturbations depends on $Bi^2 Fo = h^2/\omega k\rho c$. It is now clear that the relevant thermophysical property of the wall is the product $k\rho c$, as opposed to the conductivity alone. The effect of thermal capacity ρc is as would be expected: small thermal capacity yields fixed heat flux because all of the heat entering at the outer surface must instantaneously conduct through to the inner surface, whereas large thermal capacity of course yields fixed temperature. Some additional discussion of the physical realization of boundary conditions for the temperature perturbations is given by Sparrow *et al.* [10].

The most important conclusion of this exercise is that the boundary condition for the temperature perturbations need not be of the same type as the boundary condition for the base temperature, because of the influence of conduction within the bounding walls. Regardless of whether the base temperature has a specified heat flux or a specified wall temperature boundary condition, the boundary condition for the temperature perturbarions depends on the quantity $Bi^2 Fo$.

The literature concerning the effect of the boundary condition for the temperature perturbation will now be reviewed. Gershuni and Zhukhovitskii [4] repeated the analysis of vertical plates with a constant vertical temperature gradient, for the fixed flux boundary condition. It is shown that in both the fixed temperature and fixed flux cases, the critical wavenumber is zero, but that the critical Rayleigh number is finite for the fixed temperature case and zero for the fixed flux case. Wooding [12] and Edwards [13] showed the same result. This result is in agreement with the premise of this paper, namely that the critical Rayleigh number is zero for fixed flux boundary conditions. Yih [14] examined the same problem, but erroneously reported a nonzero Rayleigh number, as explained by Wooding [12]. Catton and Edwards [15] performed an experimental investigation concerning the flow in vertical cylinders of hexagonal cross-section, with adverse temperature gradients. They found that the critical Rayleigh number decreases with decreasing wall conductivity. However, the critical Rayleigh number would not be expected to go to zero even in the limit of zero wall conductivity, because perturbations of zero wavenumber could not occur in these finite channels.

Gershuni and Zhukhovitskii also summarized results concerning the stability of a horizontal layer between walls of arbitrary conductivity. It is well known that for the case of infinite wall conductivity (fixed temperature case), the critical Rayleigh number (based on temperature difference between the plates) is 1708, and the critical wavenumber is 3.117. As the wall conductivity decreases, the critical Rayleigh number and wavenumber both decrease. In the limiting case of zero wall conductivity, i.e. the fixed flux case, the critical wavenumber goes to zero, while the critical Rayleigh number was seen to remain finite. The same results were found earlier by Sparrow *et al.* [10], who considered the similar situation of a horizontal layer with convective thermal boundary conditions. Fixed temperature and fixed flux boundary conditions are limiting cases in this more general formulation. It is important to note that both Gershuni and Zhukhovitskii's [4] and Sparrow's [10] results, namely that the critical Rayleigh number remains finite for the fixed flux boundary condition, appear to contradict the results of the present study. The apparent discrepancy is actually just a difference of perspective, which will be explained in a later section.

PROBLEM STATEMENT

The specific physical situation under consideration is shown in Fig. 1. Two infinite parallel plates are inclined arbitrarily with respect to gravity. It is intended to consider as general a base flow as possible. The only restrictions are as follows:

(1) Only steady, parallel base flows will be considered, that is the base velocity may depend only on y, the coordinate direction normal to the plates. This is the usual fully developed assumption.

(2) There is a zero normal velocity boundary condition on at least one of the plates. Then from the parallel flow assumption above, and continuity, it follows that the y-direction base velocity, $\bar{v}$, is zero.

(3) The base temperature may vary in all three coordinate directions, but it is restricted to vary linearly in the directions parallel to the plates. In addition, the case under consideration here is that of an adverse longitudinal temperature gradient (to be defined shortly).

Now, without loss of generality, the x-direction is chosen to be in the direction of the flow, so that the base velocity is of the following form (with boldface indicating a vector quantity, and an overbar indicating the base flow):

$$\bar{\mathbf{u}} = \bar{u}(y)\mathbf{e}_x. \quad (1)$$

Note that there may be components of gravity in each of the three coordinate directions, i.e.

$$\mathbf{g} = g_x\mathbf{e}_x + g_y\mathbf{e}_y + g_z\mathbf{e}_z. \quad (2)$$

According to the third assumption above, the base temperature is given by

$$\bar{T}^*(x^*, y^*, z^*) = Ax^* + Bz^* + F(y^*). \quad (3)$$

In this equation, the temperature and coordinates are dimensional quantities (indicated with asterisks), so as to define the dimensional temperature gradients A and B, which have units of K m^{-1}. The phrase 'adverse longitudinal temperature gradient' is defined to mean that the temperature gradient parallel to the plates, projected into the direction of gravity, is positive. This is expressed mathematically as follows:

$$\mathbf{g}\cdot\nabla_{xz}\bar{T} \geqslant 0 \quad \text{where} \quad \nabla_{xz} = \frac{\partial}{\partial x}\mathbf{e}_x + \frac{\partial}{\partial z}\mathbf{e}_z \quad (4)$$

or

$$g_xA + g_zB \geqslant 0. \quad (5)$$

Notice that if the plates are horizontal, the equals sign necessarily applies, since the longitudinal temperature gradient is perpendicular to gravity.

Other than the restrictions just described, the base velocity and temperature distributions are arbitrary, and many base flow solutions which satisfy these restrictions have been published (see, for example, refs. [4, 16–18]). The flow may be purely natural convection, or it may be mixed convection driven by a pressure gradient or a moving plate. There may be a zero or nonzero net mass flow in the x-direction. There may or may not be a temperature difference between the two plates. The requirement that temperature vary linearly in the x- and z-directions may be satisfied with a variety of thermal boundary conditions for the base flow, such as specified linearly varying wall temperatures (with the same x- and z-dependence top and bottom), or specified uniform heat flux top and bottom (with top and bottom heat fluxes allowed to be different). (Note that if there is zero net mass flow, then the net heat flux must also be zero to allow a steady base solution.) There may be sources or sinks of energy and momentum (in addition to the buoyancy force), provided that they are independent of the velocity, temperature, and pressure. Note that the reasonably general problem considered here encompasses as limiting cases both horizontal and vertical geometries, and both pure natural and pure forced convection. The most severe restriction is that there must be an adverse (or zero) longitudinal temperature gradient, as given by equation (5). The case of a favorable temperature gradient is not considered here.

ANALYSIS

The linearized disturbance equations are derived as usual by expressing the velocities, temperature, and pressure as the sum of a base quantity (indicated with an overbar) and a small perturbation (indicated with a tilde), e.g. $u = \bar{u} + \tilde{u}$. These expressions are substituted into the continuity, Navier–Stokes, and energy equations (assuming the Boussinesq approximation holds and neglecting viscous dissipation in the energy equation). In these equations, the velocities are made nondimensional with any convenient characteristic velocity, U_{ch}, and the coordinates are made nondimensional with the channel spacing L. Then time and pressure are made nondimensional with L/U_{ch} and ρU_{ch}^2, respectively. The temperature is made nondimensional with the quantity γL, where γ is the longitudinal temperature gradient, projected into the direction of gravity, i.e. $\gamma = (g_x A + g_z B)/g$. Then the nondimensional temperature gradients are $\partial\bar{T}/\partial x = A/\gamma$, and $\partial\bar{T}/\partial z = B/\gamma$. Next, the terms involving only base quantities are eliminated by virtue of the fact that they satisfy the steady continuity, Navier–Stokes, and energy equations. Then, terms quadratic in the perturbations are neglected. Finally, the perturbations are expressed in the form

$$\tilde{f} = \hat{f}(y)\exp(\mathrm{i}k_x x + \mathrm{i}k_z z + \hat{\sigma}t) \tag{6}$$

where f stands for one of u, v, w, p, or T, the quantities k_x and k_z are the wavenumbers in the x- and z-directions, and $\hat{\sigma}$ is the growth rate of the disturbances. The resulting nondimensional disturbance equations are:

continuity

$$\mathrm{i}k_x\hat{u} + \hat{v}' + \mathrm{i}k_z\hat{w} = 0 \tag{7}$$

x-momentum

$$\hat{\sigma}\hat{u} + \mathrm{i}k_x\bar{u}\hat{u} + \hat{v}\bar{u}' = -\mathrm{i}k_x\hat{p} + \frac{1}{Re}(\hat{u}'' - k^2\hat{u}) - \frac{Gr}{Re^2}\frac{g_x}{g}\hat{T} \tag{8}$$

y-momentum

$$\hat{\sigma}\hat{v} + \mathrm{i}k_x\bar{u}\hat{v} = -\hat{p}' + \frac{1}{Re}(\hat{v}'' - k^2\hat{v}) - \frac{Gr}{Re^2}\frac{g_y}{g}\hat{T} \tag{9}$$

z-momentum

$$\hat{\sigma}\hat{w} + \mathrm{i}k_x\bar{u}\hat{w} = -\mathrm{i}k_z\hat{p} + \frac{1}{Re}(\hat{w}'' - k^2\hat{w}) - \frac{Gr}{Re^2}\frac{g_z}{g}\hat{T} \tag{10}$$

energy

$$\hat{\sigma}\hat{T} + \mathrm{i}k_x\bar{u}\hat{T} + \hat{u}\frac{A}{\gamma} + \hat{v}\frac{\partial\bar{T}}{\partial y} + \hat{w}\frac{B}{\gamma} = \frac{1}{Re\,Pr}(\hat{T}'' - k^2\hat{T}) \tag{11}$$

where $k^2 = k_x^2 + k_z^2$ and a prime denotes differentiation with respect to y. The Reynolds number is given by $Re = U_{\mathrm{ch}}L/\nu$, the Grashof number is defined as $Gr = g\beta\gamma L^4/\nu^2$, and Pr is the Prandtl number, ν/α.

The boundary conditions are (for walls at $y = \pm\frac{1}{2}$):

$$\hat{u}(\pm\tfrac{1}{2}) = \hat{v}(\pm\tfrac{1}{2}) = \hat{w}(\pm\tfrac{1}{2}) = 0 \tag{12}$$

and for the thermal boundary condition of fixed heat flux:

$$\frac{\partial\hat{T}}{\partial y}(\pm\tfrac{1}{2}) = 0. \tag{13}$$

Some consideration will also be given to the fixed temperature boundary condition:

$$\hat{T}(\pm\tfrac{1}{2}) = 0. \tag{14}$$

It should be noted that:

(1) There is no Squire's transformation for these equations because of the longitudinal temperature gradients A and B in the energy equation [equation (11)].

(2) The usual proof of exchange of stabilities (see for instance, Gershuni and Zhukhovitskii [4]) for the case of natural convection with the temperature increasing downward does not hold because of the nonzero base flow.

As stated previously, the goal of this paper is to show that in the case of a fixed heat flux boundary condition, the flow is unstable for any (positive) value of the Rayleigh number, that is, the critical Rayleigh number is zero. This will be demonstrated *not* by investigating all possible solutions for the perturbations, but simply by showing that the base flow is unstable to *one* class of perturbations, namely those having both wavenumbers k_x and k_z going to zero. It will also be shown that this one class of perturbations yields a nonzero critical Rayleigh number in the case of a fixed temperature boundary condition. This does not prove that the fixed temperature case is stable below this value of the Rayleigh number, however, since the flow could still be unstable to a perturbation with k_x or k_z nonzero.

Setting k_x and k_z equal to zero, the continuity equation [equation (7)] along with the boundary conditions for $\hat{v}$ [equation (12)] imply $\hat{v}(y) = 0$, and the remaining disturbance equations [equations (8)–(11)] reduce to the following:

$$\hat{\sigma}\hat{u} = \frac{1}{Re}\hat{u}'' - \frac{Gr}{Re^2}\frac{g_x}{g}\hat{T} \tag{15}$$

$$0 = -\hat{p}' - \frac{Gr}{Re^2}\frac{g_y}{g}\hat{T} \tag{16}$$

$$\hat{\sigma}\hat{w} = \frac{1}{Re}\hat{w}'' - \frac{Gr}{Re^2}\frac{g_z}{g}\hat{T} \tag{17}$$

$$\hat{\sigma}\hat{T} + \hat{u}\frac{A}{\gamma} + \hat{w}\frac{B}{\gamma} = \frac{1}{Re\,Pr}\hat{T}''. \tag{18}$$

Notice that these equations are now independent of the base flow, except for the constant temperature gradients A and B. Furthermore, as a consequence, the solutions for the disturbances *will* satisfy exchange of stabilities when there is an adverse temperature gradient (positive Gr or Ra). The y-momentum equation yields the pressure disturbance once the temperature disturbance is known, but otherwise can be removed from consideration.

Next, equations (15) and (17) are combined by multiplying them by A and B, respectively, and adding. Defining $\psi = Re(\hat{u}A + \hat{w}B)/\gamma$:

$$\hat{\sigma}\psi = \frac{1}{Re}\psi'' - \frac{Gr}{Re}\hat{T}. \tag{19}$$

Comparing the differential equations satisfied by $\hat{u}$, $\hat{w}$, and ψ (and noting that they all satisfy the same boundary conditions of vanishing at the boundaries), it is clear that $\hat{u} = (\psi/Re)(g_x/g)$, $\hat{w} = (\psi/Re)(g_z/g)$, so that $\psi^2 = Re^2(\hat{u}^2+\hat{w}^2)g^2/(g_x^2+g_z^2)$. In other words, ψ is proportional to the magnitude of the disturbance velocity vector. Now, defining $\theta = \hat{T}/Pr$, $\sigma = \hat{\sigma}\ Re\ Pr$, and $Ra = Gr\ Pr$, equations (19) and (18) become:

$$\psi'' - \frac{\sigma}{Pr}\psi - Ra\,\theta = 0 \tag{20}$$

$$\theta'' - \sigma\theta - \psi = 0. \tag{21}$$

The boundary conditions are:

$$\psi(\pm\tfrac{1}{2}) = 0 \tag{22}$$

and, for fixed heat flux boundary condition:

$$\theta'(\pm\tfrac{1}{2}) = 0 \tag{23}$$

or, for fixed temperature boundary condition:

$$\theta(\pm\tfrac{1}{2}) = 0. \tag{24}$$

These are precisely the disturbance equations for natural convection between vertical parallel plates with a longitudinal temperature gradient and zero base flow, given the same assumption that the disturbances have zero wavenumber periodicity in the x- and z-directions. These simplified equations are known to satisfy exchange of stabilities (for positive Rayleigh number), i.e. σ must be real. The general solution is:

$$\psi = a\sin\lambda_1 y + b\cos\lambda_1 y + c\sin\lambda_2 y + d\cos\lambda_2 y \tag{25}$$

$$\theta = -\frac{1}{Ra}\left[\left(\lambda_1^2+\frac{\sigma}{Pr}\right)(a\sin\lambda_1 y + b\cos\lambda_1 y) + \left(\lambda_2^2+\frac{\sigma}{Pr}\right)(c\sin\lambda_2 y + d\cos\lambda_2 y)\right] \tag{26}$$

where

$$\lambda_{1,2} = \left[-\sigma\frac{(Pr+1)}{2Pr} \pm\sqrt{\left(\sigma^2\frac{(Pr-1)^2}{4Pr^2}+Ra\right)}\right]^{1/2}. \tag{27}$$

The case of a fixed flux boundary condition will be considered first. Applying the four boundary conditions, equations (22) and (23), to the general solutions, equations (25) and (26), yields the following even and odd solutions (defined up to an arbitrary multiplicative constant). Even solution:

$$\psi = \cos\frac{\lambda_2}{2}\cos\lambda_1 y - \cos\frac{\lambda_1}{2}\cos\lambda_2 y \tag{28}$$

$$\theta = \frac{1}{Ra}\left[-\left(\lambda_1^2+\frac{\sigma}{Pr}\right)\cos\frac{\lambda_2}{2}\cos\lambda_1 y + \left(\lambda_2^2+\frac{\sigma}{Pr}\right)\cos\frac{\lambda_1}{2}\cos\lambda_2 y\right] \tag{29}$$

with σ a root of

$$\lambda_1\left(\lambda_1^2+\frac{\sigma}{Pr}\right)\sin\frac{\lambda_1}{2}\cos\frac{\lambda_2}{2} - \lambda_2\left(\lambda_2^2+\frac{\sigma}{Pr}\right)\cos\frac{\lambda_1}{2}\sin\frac{\lambda_2}{2} = 0. \tag{30}$$

Odd solution:

$$\psi = \sin\frac{\lambda_2}{2}\sin\lambda_1 y - \sin\frac{\lambda_1}{2}\sin\lambda_2 y \tag{31}$$

$$\theta = \frac{1}{Ra}\left[-\left(\lambda_1^2+\frac{\sigma}{Pr}\right)\sin\frac{\lambda_2}{2}\sin\lambda_1 y + \left(\lambda_2^2+\frac{\sigma}{Pr}\right)\sin\frac{\lambda_1}{2}\sin\lambda_2 y\right] \tag{32}$$

with σ a root of

$$\lambda_1\left(\lambda_1^2+\frac{\sigma}{Pr}\right)\cos\frac{\lambda_1}{2}\sin\lambda_2 - \lambda_2\left(\lambda_2^2+\frac{\sigma}{Pr}\right)\sin\frac{\lambda_1}{2}\cos\frac{\lambda_2}{2} = 0. \tag{33}$$

For a given Ra and Pr, equation (27) relates λ_1 and λ_2 to σ, and equations (30) and (33) can be solved for the eigenvalues σ. The solutions will be discussed in the following section of the paper.

The fixed temperature boundary condition will now be considered. Applying the four boundary conditions, equations (22) and (24), to the general solutions, equations (25) and (26), yields the following even and odd solutions (defined up to an arbitrary multiplicative constant). Even solution:

$$\psi = \cos n\pi y, \quad n \text{ odd} \tag{34}$$

$$\theta = -\frac{1}{Ra}\left(n^2\pi^2+\frac{\sigma}{Pr}\right)\cos n\pi y, \quad n \text{ odd}. \tag{35}$$

Odd solution:

$$\psi = \sin n\pi y, \quad n \text{ even} \tag{36}$$

$$\theta = -\frac{1}{Ra}\left(n^2\pi^2+\frac{\sigma}{Pr}\right)\sin n\pi y, \quad n \text{ even}. \tag{37}$$

For both the even and odd solutions, the eigenvalues σ are given by

$$\sigma = -n^2\pi^2\frac{(Pr+1)}{2} \pm\sqrt{\left(n^4\pi^4\frac{(Pr-1)^2}{4}+Ra\,Pr\right)}. \tag{38}$$

RESULTS AND DISCUSSION

The flow will be stable to the type of perturbations considered here if all eigenvalues are negative, and unstable if any eigenvalue is positive. Thus, to determine whether the flow is stable or unstable, it suffices to examine the largest eigenvalue. For the fixed flux case, the largest eigenvalue is found numerically from

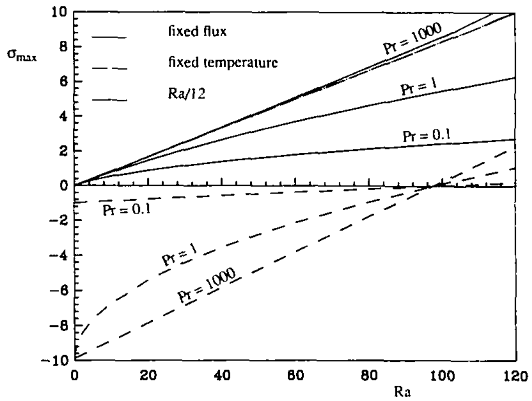

FIG. 3(a). Growth rate as a function of Rayleigh number (for small Ra).

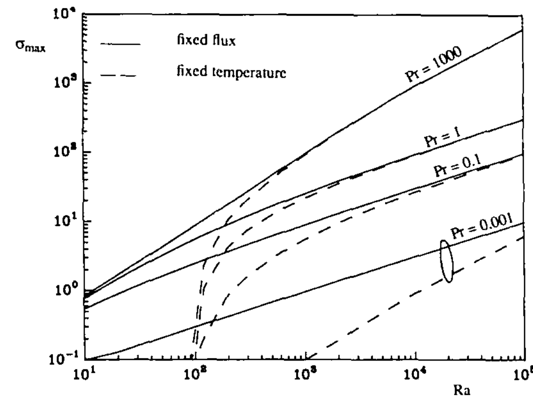

FIG. 3(b). Growth rate as a function of Rayleigh number (for large Ra).

equation (30) as a function of Ra and Pr, and is shown as the solid line in Figs. 3(a) (for small Ra) and (b) (for large Ra). This line shows how the growth rate of the most dangerous perturbation depends on the Rayleigh number. The most important thing to notice is that for all values of Ra, there exists a positive eigenvalue σ_{max}. That is, the flow is unstable to the type of perturbations considered here, all the way down to zero Rayleigh number, for which the flow is neutrally stable.

Also shown in Fig. 3(a) is the small Ra approximation for σ_{max}, for the fixed flux case (dash–dot line). The small Ra approximation is found by expanding equations (27) and (30) for small Ra, which implies small λs and σ_{max}. It is given by

$$\lim_{Ra\to 0} \sigma_{max} = \frac{Ra}{12}. \tag{39}$$

It can be seen that this agrees very well with the exact solution for large Pr, even for Ra on the order of 100. For small values of the Prandtl number, the exact curves deviate from the approximation at lower values of Ra.

For large Rayleigh number, it can be seen from equation (27) that λ_2 is large and imaginary. This enables simplifications to equation (30), which results in $\lambda_1 \tan(\lambda_1/2)$ going to infinity as Ra goes to infinity. The solution is $\lambda_1 = n\pi$, n odd. Then σ is given by the same formula as for the fixed temperature case [equation (38)]. In particular, the largest eigenvalue σ_{max} is given by equation (40) below. Thus, for large Rayleigh number, σ_{max} for the fixed flux case approaches the fixed temperature result. This is confirmed in Fig. 3(b).

For the fixed temperature case, the largest value of σ is found from equation (38), with $n = 1$, and the positive sign:

$$\sigma_{max} = -\pi^2\frac{(Pr+1)}{2} + \sqrt{\left(\pi^4\frac{(Pr-1)^2}{4} + Ra\,Pr\right)}. \tag{40}$$

This is also shown in Figs. 3(a) and (b) (dashed line). Clearly, σ_{max} is only positive provided $Ra > \pi^4$. Therefore, for the fixed temperature boundary condition, the flow is stable to the type of perturbations considered here, for $Ra < \pi^4$. This does not prove that the flow is truly stable, since it may be unstable to a finite wavelength perturbation. In general, for the case of a fixed temperature boundary condition, the stability of the flow will depend on the base flow and inclination.

Some small insight into the reason that the fixed flux case is neutrally stable for $Ra = 0$ while the fixed temperature case is stable (to the type of perturbations considered here), can be gained by considering the disturbance equations, equations (20) and (21), for $Ra = 0$. In the case of fixed temperature, $\sigma = 0$ is not an admissible eigenvalue, because it yields only the trivial solution, $\psi = \theta = 0$. In contrast, for the case of fixed flux, $\sigma = 0$ yields the nontrivial solution, $\psi = 0$, $\theta = \text{const.}$ In other words, since the fixed flux case admits a temperature perturbation of the form $\theta = \text{const.}$ at zero Rayleigh number, it is only neutrally stable at $Ra = 0$, and becomes unstable for infinitesimally small Rayleigh number.

It should be recalled that the previous analyses of Sparrow *et al.* [10] and Gershuni and Zhukhovitskii [4] for horizontal plates with fixed flux boundary conditions yielded a nonzero critical Rayleigh number, appearing to contradict the present study. This apparent discrepancy will now be explained. First, it should be noted that the Rayleigh number in the earlier studies was based on the temperature difference between the plates ($Ra_{\Delta T}$), whereas the Rayleigh number in this study (Ra) is based on the longitudinal temperature gradient projected into the direction of gravity (i.e. γ). Based on this definition, the horizontal case has $Ra = 0$, so according to the present study the flow should be neutrally stable (or unstable to some other type of perturbation than is considered here) for any value of $Ra_{\Delta T}$. However, the authors of the earlier studies reported a critical $Ra_{\Delta T}$ of 720. The discrepancy is simply a matter of perspective: the objective of those studies was to find the critical Rayleigh number *for the onset of motion* (from a quiescent base flow). To this end, both previous analyses searched for nonzero solutions for the vertical velocity perturbation (v in the present notation). The solution $\tilde{v} = 0$ was not considered, since it is a quiescent solution. On the contrary, in the present study the solution $\tilde{v} = \tilde{u} = 0$, $\tilde{T} =$ const. defines a perturbation to which the base solution is neutrally stable; if the temperature is perturbed by a fixed amount, the new temperature profile will persist rather than returning to the original solution.

Energy integrals

Many authors have interpreted their linear stability results by evaluating the energy integrals associated with the velocity and temperature disturbances. Joseph [19] gives a thorough derivation of these energy integrals. An analysis of the energy integrals indicates what the source of the instability is, for example, streamwise buoyancy, cross-stream buoyancy, mean shear. For the type of perturbations considered here, these integrals are particularly simple, and show that the only source of the instability is streamwise buoyancy.

In the interest of simplicity, the energy integrals will be derived here beginning with the most simplified form of the equations, equations (20) and (21). Since exchange of stabilities holds for these equations, i.e. σ is real, ψ and θ are real functions. Thus, there is no need to use complex conjugates in deriving the energy integrals. Equations (20) and (21) are multiplied by ψ and θ, respectively, and integrated from $y = -\frac{1}{2}$ to $\frac{1}{2}$ to yield the following equations. Recall that ψ^2 is proportional to $(\hat{u}^2 + \hat{w}^2)$, so that ψ^2 represents the kinetic energy of the disturbances. The integral equation associated with the temperature disturbances does not actually represent a form of energy.

$$\frac{\sigma}{Pr}\int_{-1/2}^{1/2} \psi^2 \, \mathrm{d}y = -\int_{-1/2}^{1/2} \psi'^2 \, \mathrm{d}y - Ra \int_{-1/2}^{1/2} \psi\theta \, \mathrm{d}y \tag{41}$$

$$\sigma \int_{-1/2}^{1/2} \theta^2 \, \mathrm{d}y = -\int_{-1/2}^{1/2} \theta'^2 \, \mathrm{d}y - \int_{-1/2}^{1/2} \psi\theta \, \mathrm{d}y. \tag{42}$$

Use has been made of the identity $\int ff'' \, \mathrm{d}y = -\int f'^2 \, \mathrm{d}y$, provided that either f or f' is zero at both boundaries, which is true for both ψ and θ.

In the first of these equations, the left-hand side corresponds to the rate of change of kinetic energy of the disturbances, the first term on the right-hand side represents viscous dissipation, and the last term corresponds to streamwise buoyancy. Since the first integral on the right-hand side is always positive, viscous dissipation always reduces the kinetic energy, and is therefore stabilizing. The only possible source of kinetic energy (for the type of perturbations considered here) is buoyancy in the flow direction. The thermal 'energy' equation contains similar dissipation and source terms, and demonstrates the same point. Thus, the physical mechanism of instability is the well-known phenomenon that if a 'particle' of fluid is displaced in the upward flow direction, it will encounter cooler fluid, and if it is displaced in the downward flow direction it will encounter warmer fluid. In either case, the buoyancy of the particle relative to its surroundings will make it tend to continue in the same direction.

It is interesting to consider equations (41) and (42) for $Ra = 0$. In equation (41), it is clear that σ must be negative (i.e. stable flow), unless $\int \psi'^2 \, \mathrm{d}y = 0$. This is only possible if $\psi' = 0$, which implies $\psi = 0$, given the boundary conditions. Now in equation (42), it is clear that σ must be negative unless $\int \theta'^2 \, \mathrm{d}y = 0$. This is only possible if $\theta' = 0$. In the fixed temperature case this yields the trivial solution, $\theta = 0$, but in the fixed flux case it yields the nontrivial solution $\theta =$ const., as discussed earlier. That is, for the $Ra = 0$ case, as long as θ' is nonzero, dissipation causes the flow to be stable. The flow can only be neutrally stable if $\theta' = 0$, which is allowed by the fixed flux case.

CONCLUSIONS

This paper concerns the linear stability of mixed or natural convection between parallel plates inclined arbitrarily with respect to gravity. The base velocity and temperature distributions are allowed to be quite general. The only restrictions are that (1) the base flow must be steady and parallel, (2) there must be a zero normal velocity boundary condition on at least one plate, (3) the temperature can vary at most linearly in the directions parallel to the plates, and (4) there is an adverse (or zero) longitudinal temperature gradient [defined by equation (5)]. It was shown that, if the boundary condition for the temperature perturbations is fixed flux, then *any* such flow is unstable for all (positive) values of the Rayleigh number (where the Rayleigh number is based on the longitudinal temperature gradient projected in the direction of gravity). That is, the critical Rayleigh number is zero. This result was demonstrated by investigating the stability of the flow to perturbations with zero wavenumber in the directions parallel to the plates. The implicit equation for the maximum growth rate [equation (30)] was solved numerically over the range $0 \leqslant Ra \leqslant 10^5$. The large and small Rayleigh number approximations were also given [equations (39) and (40)]. It was found that there is always a positive growth rate for any positive Rayleigh number, with the growth rate going to zero as Rayleigh goes to zero. By examining the energy integrals, it was seen that the only source of energy for these disturbances is buoyancy in the flow direction.

This result may have significance in low Rayleigh number natural and mixed convection applications, such as solar collectors and chemical vapor deposition systems. Depending on whether instability is desirable or undesirable, the walls might be constructed to approximate either fixed heat flux or fixed temperature. Of course, any real system is of finite size, which prohibits the wavenumbers from going to zero. None the less, this work does indicate that fixed flux boundary conditions tend to promote instability more than do fixed temperature boundary conditions.

To conclude, an entire class of flows has been shown to be unstable down to zero Rayleigh number. It is therefore not necessary to independently determine the neutral stability boundary for any subset of this class.

Acknowledgements—The author greatly appreciated many helpful discussions with Professors Robert Kelly, Arne Pearlstein, and Ivan Catton, and the constructive comments of the reviewers.

REFERENCES

1. I. Catton, Natural convection in enclosures, *Proc. 6th Int. Heat Transfer Conf.*, Vol. 6, pp. 13–31 (1978).
2. S. Ostrach, Laminar flows with body forces. In *Theory of Laminar Flows* (Edited by F. K. Moore), Section F. Princeton University Press, Princeton, New Jersey (1964).
3. S. Ostrach, Natural convection in enclosures, *J. Heat Transfer* **110,** 1175–1190 (1988).
4. G. Z. Gershuni and E. M. Zhukhovitskii, *Convective Stability of Incompressible Fluids*, in translation, Keter, Jerusalem (1972, translation 1976).
5. C. M. Vest and V. S. Arpaci, Stability of natural convection in a vertical slot, *J. Fluid Mech.* **36**(1), 1–15 (1969).
6. J. E. Hart, Stability of the flow in a differentially heated inclined box, *J. Fluid Mech.* **47**(3), 547–576 (1971).
7. R. F. Bergholz, Instability of steady natural convection in a vertical fluid layer, *J. Fluid Mech.* **84**(4), 743–768 (1978).
8. W. Nakayama, G. J. Hwang and K. C. Cheng, Thermal instability in plane Poiseuille flow, *J. Heat Transfer* **92,** 61–68 (1970).
9. A. Balakrishnan, Thermal instability in Poiseuille flow with axial temperature gradient, Engineering 231A Report to Prof. I. Catton, University of California, Los Angeles (1972).
10. E. M. Sparrow, R. J. Goldstein and V. K. Jonsson, Thermal instability in a horizontal fluid layer: effect of boundary conditions and non-linear temperature profile, *J. Fluid Mech.* **18,** 513–528 (1964).
11. C. P. Knowles and B. Gebhart, The stability of the laminar natural convection boundary layer, *J. Fluid Mech.* **34**(4), 657–686 (1968).
12. R. A. Wooding, Instability of a viscous liquid of variable density in a vertical Hele-Shaw cell, *J. Fluid Mech.* **7**(4), 501–515 (1960).
13. D. K. Edwards, Suppression of cellular convection by lateral walls, *J. Heat Transfer* **91,** 145–150 (1969).
14. C. S. Yih, Thermal instability of viscous fluids, *Q. Appl. Math.* **17**(1), 25–42 (1959).
15. I. Catton and D. K. Edwards, Effect of side walls on natural convection between horizontal plates heated from below, *J. Heat Transfer* **89,** 295–299 (1967).
16. A. S. Lavine, Analysis of fully developed opposing mixed convection between inclined parallel plates, *Wärme- und Stoffübertragung* **23,** 249–257 (1988).
17. S. Ostrach, Combined natural- and forced-convection laminar flow and heat transfer of fluids with and without heat sources in channels with linearly varying wall temperatures, NACA Technical Note 3141, pp. 1–74 (1954).
18. T. T. Hamadah and R. A. Wirtz, Analysis of laminar fully developed mixed convection in a vertical channel with opposing buoyancy, *J. Heat Transfer* **113,** 507–510 (1991).
19. D. D. Joseph, *Stability of Fluid Motions*, Vol. II. Springer, New York (1976).